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The added-mass coefficients of a torus

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SUMMARY

The generalised added-mass coefficients of a torus in translatory and rotational motion in an inviscid incompressible fluid are obtained via an exact solution of Laplace's equation in toroidal coordinates. Of the six possible independent coefficients three are found to have nonzero, finite and separate values, due to symmetry. These are translation in, and perpendicular to the ring plane and rotation around a diameter. For translation normal to the ring plane, the added mass is somewhat larger than the mass of the torus of equal density. This coefficient tends to the torus mass for slender tori (large ratio of ring to core diameters). For translation in the ring plane the added mass tends to one half the torus mass, and for rotation the added inertia is approximately the torus moment of inertia for such slender tori. Simple relations for the added-mass coefficients as a function of the diameter ratio for general tori are also presented.

1. Introduction

Study of the motion of toroidal shapes in a fluid medium is of interest from both the fundamental and practical points of view. The torus is one of the most elementary non-simply connected bodies and as such, poses some new mathematical requirements. Physically the torus is a close approximation of the shape of the fluid body attached to a vortex ring resulting, for example, from the entrance of a fluid jet of finite duration into a fluid medium of comparable density. Such toroidal shapes have been observed in intermittent operation of smokestacks [1], when liquid drops enter a container of the same liquid [2], when shock waves pass through a medium with a sudden change in density [3] and in superfluids [4].

For this reason it is important to have an estimate of the added mass coefficients of toroidal forms in analysing various modes of vortex-ring motion. The added-mass coefficients may be considered as a measure of the fluid energy excited by the motion of the body in an otherwise quiescent fluid medium and is a fundamental parameter in the analysis of the kinematics of Rankine bodies.

While discussions of vortex-ring motions have been available for over a century starting from Helmholtz's classical work, the added mass of the torus, to the best knowledge of the authors, has not yet been calculated, except for an asymptotic analysis for tori of very large ratios of ring-to-core radii, presented recently by Wu and Yates [12]. The purpose of the present paper is to establish the values of the added-mass coefficient in translational and rotational motion for a general torus.

The analysis is carried out by assuming the fluid to be inviscid and incompressible, and in terms of toroidal harmonics, first introduced by Hicks [5] and Dyson [6]. The three distinct

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added-mass coefficients of the torus corresponding to axial and transverse linear motions and rotation about the transverse axis are computed from the expressions of the kinetic energy. The three added mass coefficients are found from the associated three Kirchhoff's velocity potentials and use is made of the orthogonality properties of the toroidal harmonics. For the axisymmetric case, an alternative derivation for the corresponding added mass which is based on the employment of a stream function is also given. The solution for the added-mass coefficients is obtained in the form of a Neumann series with coefficients obtained by the solution of a tri-diagonal matrix. Numerical results for the three added masses are also presented for various ring geometries and some simple practical expressions for these coefficients in terms of the ring and core radii are also given.

2. Toroidal coordinates and boundary conditions

Let the origin of a Cartesian coordinate system (x, y, z) be placed at the center of a torus of radius d and core radius b such that the x axis coincides with the axis of symmetry. When dealing with toroidal geometries it is more convenient to employ a toroidal orthogonal coordinate system (τ, β, γ) that is related to the Cartesian system by the following transformation [7]:

$$x = \frac{a \sin \beta}{\cosh \tau - \cos \beta}; \quad y = \frac{a \sinh \tau \cos \gamma}{\cosh \tau - \cos \beta}; \quad z = \frac{a \sinh \tau \sin \gamma}{\cosh \tau - \cos \beta}, \tag{1}$$

where $0 \leqq \tau < \infty$, $-\pi \leqq \beta \leqq \pi$, $0 \leqq \gamma \leqq 2\pi$, and a is a characteristic parameter. The metrics of the above transformation are given by

$$g_{\beta\beta} = g_{\tau\tau} = \frac{a^2}{(\cosh \tau - \cos \beta)^2}, \quad g_{\gamma\gamma} = \frac{a^2 \sinh^2 \tau}{(\cosh \tau - \cos \beta)^2}, \tag{2}$$

and the geometrical parameters of the torus are (Fig. 1)

$$d = a \coth \tau_0, \quad b = a \operatorname{cosech} \tau_0, \quad V_T = 2\pi^2 a^3 \coth \tau_0 \operatorname{cosech}^2 \tau_0, \tag{3}$$

where $\tau = \tau_0 =$ const. is the equation of the toroidal surface and V_T denotes the volume of the torus.

In the present analysis we assumed the fluid medium to be inviscid and incompressible which implies the existence of a velocity-potential function governed by the Laplace equation. A normal separable solution of the Laplace equation in toroidal coordinates which vanishes at infinity (exterior harmonics) is given by

$$\Phi(\tau, \beta, \gamma) = \sqrt{2 \cosh \tau - 2 \cos \beta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P^m_{n-1/2}(\cosh \tau) \begin{matrix} \sin(n\beta) & \sin(m\gamma) \\ \cos(n\beta) & \cos(m\gamma) \end{matrix}. \tag{4}$$

Similarly, a normal harmonic solution which is regular at the origin (interior harmonic) may be written as

$$\phi(\tau, \beta, \gamma) = \sqrt{2 \cosh \tau - 2 \cos \beta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q^m_{n-1/2}(\cosh \tau) \begin{matrix} \sin(n\beta) & \sin(m\gamma) \\ \cos(n\beta) & \cos(m\gamma) \end{matrix}, \tag{5}$$

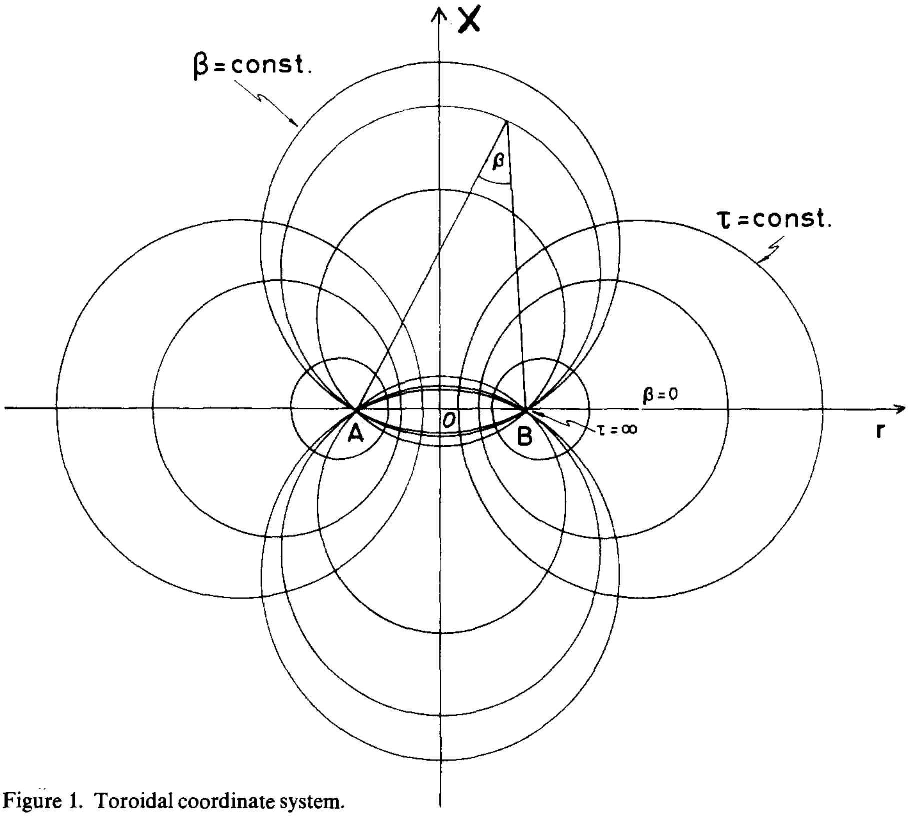

Figure 1. Toroidal coordinate system.

where $P^m_{n-1/2}$ and $Q^m_{n-1/2}$ denote the Legendre functions of first and second kind respectively.

It is further assumed that the torus is translating with velocity $V(V_1, V_2, V_3)$ and rotating with angular velocity $w(w_1, w_2, w_3)$ such that (V_1, w_1), (V_2, w_2) and (V_3, w_3) are the translatory and angular velocities along the x,y, and z directions respectively. The total velocity potential of such a motion may be expressed in terms of six unit Kirchhoff potentials:

$$\Phi(\tau, \beta, \gamma) = V_1\phi_1 + V_2\phi_2 + V_3\phi_3 + w_1\phi_4 + w_2\phi_5 + w_3\phi_6. \tag{6}$$

Denoting the radius vector and the unit normal vector, corresponding to points of the torus, by $\boldsymbol{r}$ and $\boldsymbol{n}$ respectively, the Neumann type boundary condition on the toroidal surface implies that

$$\frac{\partial \Phi}{\partial n} = (\boldsymbol{V}\cdot\boldsymbol{n}) + ([\boldsymbol{w} \times \boldsymbol{r}]\cdot\boldsymbol{n}) \tag{7}$$

which together with (6) yields

$$\frac{\partial \phi_1}{\partial n} = n_1, \quad \frac{\partial \phi_2}{\partial n} = n_2, \quad \frac{\partial \phi_3}{\partial n} = n_3,$$

$$\frac{\partial \phi_4}{\partial n} = yn_3 - zn_1, \quad \frac{\partial \phi_5}{\partial n} = zn_1 - xn_3, \quad \frac{\partial \phi_6}{\partial n} = xn_2 - yn_1, \tag{8}$$

where (n_1, n_2, n_3) are the three components of the normal vector.

The six added-mass coefficients λ_i may be expressed in terms of the kinetic energy of the fluid [8]. Denoting the added masses, corresponding to ϕ_i, by λ_i, $i = 1, 2, \ldots, 6$, one has

$$\lambda_i = \rho \int_S \phi_i \frac{\partial \phi_i}{\partial n} dS, \qquad i = 1, 2, \ldots, 6 \tag{9}$$

where ρ is the fluid density and S denotes the surface of the torus. The three values λ_1, λ_2 and λ_3 correspond to pure translation whereas λ_4, λ_5 and λ_6 correspond to pure rotation. It is clear that because of the symmetry properties of the torus,

$$\lambda_2 = \lambda_3, \quad \lambda_5 = \lambda_6, \quad \lambda_4 = 0, \tag{10}$$

thus only three finite distinct values exist, namely λ_1, λ_2 and λ_6. The three components of the unit normal vector $\boldsymbol{n}$ may be expressed as

$$n_1 = \frac{1}{\sqrt{g_{\tau\tau}}} \frac{\partial x}{\partial \tau}; \quad n_2 = \frac{1}{\sqrt{g_{\tau\tau}}} \frac{\partial y}{\partial \tau}; \quad n_3 = \frac{1}{\sqrt{g_{\tau\tau}}} \frac{\partial z}{\partial \tau} \tag{11}$$

which upon substituting in (9) implies that

$$\lambda_1 = \rho \int_0^{2\pi} \int_{-\pi}^{+\pi} \phi_1 \frac{a \sinh \tau_0}{\cosh \tau_0 - \cos \beta} \frac{\partial x}{\partial \tau} d\beta d\gamma, \tag{12}$$

$$\lambda_2 = \rho \int_0^{2\pi} \int_{-\pi}^{+\pi} \phi_2 \frac{a \sinh \tau_0}{\cosh \tau_0 - \cos \beta} \frac{\partial y}{\partial \tau} d\beta d\gamma,$$

and

$$\lambda_6 = \rho \int_0^{2\pi} \int_{-\pi}^{+\pi} \phi_6 \frac{a \sinh \tau_0}{\cosh \tau_0 - \cos \beta} \left(x \frac{\partial y}{\partial \tau} - y \frac{\partial x}{\partial \tau} \right) d\beta d\gamma. \tag{13}$$

In the following sections the above integrals are computed and explicit expressions for the three distinct added-mass coefficients of the torus are derived.

3. Added mass for longitudinal axisymmetric motion

The potential function ϕ_1, corresponding to translation with unit velocity in the direction of the axis of symmetry, satisfies the following boundary condition:

$$\frac{\partial}{\partial \tau}(\phi_1 - x) = 0 \text{ on } \tau = \tau_0 \tag{14}$$

and an additional requirement that ϕ_1 should vanish at infinity, i.e., at $\tau \to 0$. In order to solve (14) for ϕ_1 we employ the harmonic representation of x, namely,

$$x = \frac{a \sin \beta}{\cosh \tau - \cos \beta} = a\sqrt{2 \cosh \tau - 2 \cos \beta} \sum_{n=0}^{\infty} \bar{A}_n Q_{n-1/2}(\cosh \tau) \sin(n\beta), \tag{15}$$

where $\bar{A}_n$ are coefficients to be determined from the following useful relation [7]:

$$\frac{1}{\sqrt{2\cosh\tau - 2\cos\beta}} = \frac{1}{\pi}\left[Q_{-1/2}(\cosh\tau) + 2\sum_{n=1}^{\infty} Q_{n-1/2}(\cosh\tau)\cos(n\beta)\right]. \tag{16}$$

Substituting (16) into (15) renders

$$\frac{a\sin\beta}{(2\cosh\tau - 2\cos\beta)^{\frac{3}{2}}} =$$

$$= -a\frac{\partial}{\partial\beta}\left\{\frac{1}{\sqrt{2\cosh\tau - 2\cos\beta}}\right\} = \frac{2}{\pi}\sum_{n=1}^{\infty} nQ_{n-1/2}(\cosh\tau)\sin(n\beta). \tag{17}$$

Comparing (15) and (16) in view of (17) yields

$$\bar{A}_n = 4n/\pi. \tag{18}$$

The harmonic representation of ϕ_1 suggested by (14) and (15) is now expressed as

$$\phi_1(\tau,\beta) = \frac{4a}{\pi}\sqrt{2\cosh\tau - 2\cos\beta}\sum_{n=1}^{\infty} A_n P_{n-1/2}(\cosh\tau)\sin(n\beta) \tag{19}$$

where the coefficients A_n are to be found from the solution of

$$\frac{\partial}{\partial\tau}\{\sqrt{2\cosh\tau - 2\cos\beta}\sum_{n=1}^{\infty}[A_n P_{n-1/2}(\cosh\tau) - nQ_{n-1/2}(\cosh\tau)]\sin(n\beta)\}|_{\tau=\tau_0} = 0 \tag{20}$$

which is obtained by substituting (15) and (19) in the boundary condition (14). By applying the orthogonality properties to (20), the following recurrence formula for the coefficients A_n is obtained:

$$\alpha_n A_{n-1} + \beta_n A_n + \gamma_n A_{n+1} = \delta_n, \qquad n = 1, 2, \ldots \tag{21}$$

where $A_0 = 0$ and

$$\alpha_n = -\dot{P}_{n-3/2}(\cosh\tau_0) \tag{22}$$

$$\beta_n = P_{n-1/2}(\cosh\tau_0) + 2\cosh\tau_0\dot{P}_{n-1/2}(\cosh\tau_0) \tag{23}$$

$$\gamma_n = -\dot{P}_{n+1/2}(\cosh\tau_0) \tag{24}$$

$$\delta_n = -(n-1)\dot{Q}_{n-3/2}(\cosh\tau_0) + n[Q_{n-1/2}(\cosh\tau_0) + 2\cosh\tau_0\dot{Q}_{n-1/2}(\cosh\tau_0)] - (n+1)\dot{Q}_{n+1/2}(\cosh\tau_0). \tag{25}$$

Here the dot denotes differentiation with respect to the argument. The inhomogeneous second-order linear difference equation (21) yields an explicit solution by using the Thomas algorithm [9]. For $n < N$, where N is an arbitrary integer, we set $\alpha_1 = \gamma_N = 0$; then the solution of (21) in a retrogressive form is given by,

$$A_k = \delta'_k - \gamma'_k A_{n+1}, \; A_N = \delta'_N, \qquad k = N-1, N-2, \ldots, 2, 1 \tag{26}$$

where

$$\gamma'_{k+1} = \frac{\gamma_{k+1}}{\beta_{k+1} - \alpha_{k+1}\gamma'_k}, \quad \delta'_{k+1} = \frac{\delta_{k+1} - \alpha_{k+1}\delta'_k}{\beta_{k+1} - \alpha_{k+1}\gamma'_k}, \quad k = 1, 2, \ldots, N-1 \tag{27}$$

and

$$\gamma'_1 = \gamma_1/\beta_1; \quad \delta'_1 = \delta'_1/\beta_1.$$

To find the value of λ_1, both (15) and (19) are substituted into (12) which, after some manipulation, yields

$$\lambda_1 = -\frac{16\sqrt{2}}{3}\rho a^3 \sinh^2 \tau_0 \sum_{n=1}^{\infty} nA_n P_{n-1/2}(\cosh \tau_0) \int_0^{2\pi} \frac{\cos(n\beta)d\beta}{(\cosh \tau_0 - \cos \beta)^{\frac{3}{2}}}. \tag{28}$$

The following relation is easily obtained from (16)

$$Q_{n-1/2}(\cosh \tau) = \int_0^{\pi} \frac{\cos(n\beta)d\beta}{(2\cosh \tau - 2\cos \beta)^{\frac{1}{2}}} \tag{29}$$

which by differentiation with respect to the argument renders the following relation:

$$\dot{Q}_{n-1/2}(\cosh \tau) = -\int_0^{\pi} \frac{\cos(n\beta)d\beta}{(2\cosh \tau - 2\cos \beta)^{\frac{3}{2}}}. \tag{30}$$

Substituting (30) into (28) yields the desired expression for the longitudinal added-mass coefficients

$$\lambda_1 = \tfrac{128}{3}\rho a^3 \sinh^2 \tau_0 \sum_{n=1}^{\infty} nA_n P_{n-1/2}(\cosh \tau_0)\dot{Q}_{n-1/2}(\cosh \tau_0) \tag{31}$$

where the coefficients A_n are given by the solution of (21).

4. Added mass for longitudinal transverse motion

To calculate the added-mass coefficient for transverse motion in the y direction, it is advantageous to employ the harmonic representation of y which is taken here in the following form:

$$y = \frac{a \sinh \tau \cos \gamma}{\cosh \tau - \cos \beta} = a\sqrt{2\cosh \tau - 2\cos \beta} \sum_{n=0}^{\infty} \bar{B}_n Q^1_{n-1/2}(\cosh \tau)\cos(n\beta)\cos \gamma. \tag{32}$$

Equation (16) implies that the unknown coefficients in (32) are given by

$$\bar{B}_0 = -2/\pi, \quad \bar{B}_n = -4/\pi, \qquad n \geqq 1. \tag{33}$$

The above harmonic form of y and the conditions at infinity suggest that the potential ϕ_2 may be expressed as

$$\phi_2(\tau, \beta, \gamma) = \frac{2a}{\pi}\sqrt{2\cosh \tau - 2\cos \beta} \sum_{n=0}^{\infty} B_n P^1_{n-1/2}(\cosh \tau)\cos(n\beta)\cos \gamma \tag{34}$$

where B_n are coefficients to be determined.

Applying the boundary condition on the toroidal surface,

$$\frac{\partial}{\partial \tau}(\phi_2 - y) = 0 \quad \text{on} \quad \tau = \tau_0 \tag{35}$$

yields the following recurrence formula for the coefficients B_n:

$$\alpha_n B_{n-1} + \beta_n B_n + \gamma_n B_{n+1} = \delta_n \qquad n = 0, 1, \ldots, N \tag{36}$$

where $\alpha_0 = \gamma_N = 0$. Here the coefficients α_n, β_n, γ_n and δ_n are given by

$$\alpha_n = (1 + \varepsilon_{n1})\dot{P}_{n-3/2}(\cosh \tau_0), \qquad n \geqq 1, \tag{37}$$

$$\beta_n = -[P^1_{n-1/2}(\cosh \tau_0) + 2\cosh \tau_0 \dot{P}^1_{n-1/2}(\cosh \tau_0)], \qquad n \geqq 0, \tag{38}$$

$$\gamma_n = \dot{P}^1_{n+1/2}(\cosh \tau_0), \tag{39}$$

$$\delta_n = -\left(\frac{2}{1 + \varepsilon_{n0}}\right)[\dot{Q}^1_{n-3/2}(\cosh \tau_0) + \dot{Q}^1_{n+1/2}(\cosh \tau_0) - Q^1_{n-1/2}(\cosh \tau_0) - {} - 2\cosh \tau_0 \dot{Q}^1_{n-1/2}(\cosh \tau_0), \tag{40}$$

where ε_{ij} is the Kronecker symbol which is one for $i = j$ and zero otherwise. The Thomas algorithm may be used again to solve the linear difference equation (36) for the coefficients B_n in a manner similar to that described in the previous section. Substituting (32) and (34) into (12) renders the following expression for the added-mass coefficient:

$$\lambda_2 = -16\rho a^3 \sinh \tau_0 \sum_{n=0}^{\infty} B_n P^1_{n-1/2}(\cosh \tau_0)[\cosh \tau_0 \dot{Q}_{n-1/2}(\cosh \tau_0) + {} + \tfrac{2}{3}\sinh^2 \tau_0 \ddot{Q}_{n-1/2}(\cosh \tau_0)] \tag{41}$$

where the two dots denote second derivative with respect to the argument. In the derivation of (41) both the orthogonality properties of the Legendre functions and the relation (29) have also been employed.

5. Added mass for rotation about the transverse axis

The added-mass coefficient corresponding to pure rotation about the z axis is given by (13) where the unit potential ϕ_6 is an exterior toroidal harmonic satisfying

$$\frac{\partial \phi_6}{\partial \tau} = x\frac{\partial y}{\partial \tau} - y\frac{\partial x}{\partial \tau} = \frac{a^2 \cosh \tau \sin \beta \cos \gamma}{(\cosh \tau - \cos \beta)^2}, \quad \text{on} \quad \tau = \tau_0, \tag{42}$$

The boundary condition (42) and the conditions at infinity suggest the following form for the Kirchhoff potential, ϕ_6:

$$\phi_6(\tau, \beta, \gamma) = a^2\sqrt{2\cosh \tau - 2\cos \beta} \sum_{n=1}^{\infty} C_n P^1_{n-1/2}(\cosh \tau)\sin(n\beta)\cos \gamma. \tag{43}$$

Substituting (43) into (42) and employing (16) one obtains the following recurrence formulae for the coefficients C_n:

$$\alpha_n C_{n-1} + \beta_n C_n + \gamma_n C_{n+1} = \delta_n, \qquad n = 1, 2, \ldots, N, \tag{44}$$
$$\alpha_1 = \gamma_N = 0.$$

The above finite difference equation is solved by the Thomas algorithm as given in (26) with the understanding that A_k is replaced by C_k and

$$\alpha_n = -\dot{P}^1_{n-3/2}(\cosh \tau_0), \tag{45}$$

$$\beta_n = P^1_{n-1/2}(\cosh \tau_0) + 2 \cosh \tau_0 \dot{P}^1_{n-1/2}(\cosh \tau_0), \tag{46}$$

$$\gamma_n = -\dot{P}^1_{n+1/2}(\cosh \tau_0), \tag{47}$$

$$\delta_n = \frac{8n}{\pi} \coth \tau_0 Q_{n-1/2}(\cosh \tau_0). \tag{48}$$

Substituting (42) and (43) into (13) and employing (30), one obtains the following expression for the rotational added mass:

$$\lambda_6 = -\tfrac{8}{3}\pi a^5 \rho \sinh(2\tau_0) \sum_{n=1}^{\infty} n C_n P^1_{n-1/2}(\cosh \tau_0) \dot{Q}_{n-1/2}(\cosh \tau_0). \tag{49}$$

6. Calculation of λ_1 using stream function formulation

An alternative approach for the calculation of λ_1 which yields simpler results but is restricted to axisymmetric motion alone, is based on the concept of the stream function.

The axisymmetric Stokes stream function corresponding to a translation of a torus in an infinite medium with unit velocity in the direction of the axis of symmetry [10] is

$$\psi(\tau, \beta) = \tfrac{1}{2}(y^2 + z^2) + $$
$$+ \frac{\sinh \tau}{2\sqrt{2 \cosh \tau - 2 \cos \beta}} \sum_{n=0}^{\infty} (2 - \varepsilon_{n0}) D_n P^1_{n-1/2}(\cosh \tau) \cos(n\beta) \tag{50}$$

where ε_{ij} again denotes the Kronecker delta function. The coefficients D_n in (50) are given by

$$D_n = \frac{4Q^1_{n-1/2}(\cosh \tau_0)}{\pi P^1_{n-1/2}(\cosh \tau_0)} \left(a^2 + \frac{4A}{4n^2 - 1}\right). \tag{51}$$

Here the coefficient A denotes the value of the stream function on the toroidal surface, namely

$$\psi(\tau_0, \beta) = A = \text{const.} \tag{52}$$

The above boundary condition is not sufficient for the determination of the constant A since the torus is a stream surface. This is expected, due to the fact that the flow domain is multiply connected and as such does not yield a unique solution. To obtain uniqueness we impose the condition of a circulation-free flow which provides an additional condition on the stream function:

$$\int_0^{\pi} \left.\frac{\partial \psi}{\partial \tau}\right|_{\tau=\tau_0} d\beta = 0. \tag{53}$$

Applying (53) to (50) and (51) yields the following expression for A:

$$A = \left(G - \sum_{n=0}^{\infty} b_n F_n\right) \Big/ \sum_{n=0}^{\infty} d_n F_n, \tag{54}$$

where

$$b_n = 4a^2 Q^1_{n-1/2}(\cosh \tau_0)/[\pi P^1_{n-1/2}(\cosh \tau_0)], \tag{55}$$

$$d_n = 8Q^1_{n-1/2}(\cosh \tau_0)/[\pi(4n^2 - 1)P^1_{n-1/2}(\cosh \tau_0)], \tag{56}$$

$$G = \frac{\pi a^2}{2 \sinh^2 \tau_0}, \tag{57}$$

and

$$F_n = (2 - \varepsilon_{n0})\{\dot{P}^1_{n-1/2}(\cosh \tau_0)Q_{n-1/2}(\cosh \tau_0) \sinh^2 \tau_0 + \\ + P^1_{n-1/2}(\cosh \tau_0)[\cosh \tau_0 Q_{n-1/2}(\cosh \tau_0) + \sinh^2 \tau_0 \dot{Q}_{n-1/2}(\cosh \tau_0)]\}. \tag{58}$$

Having found the stream function the kinetic energy of the flow exterior to the torus E may be computed from the following relation [8]:

$$E = -\pi\rho \int_0^{\pi} \frac{\psi^*}{\sqrt{y^2 + z^2}} \cdot \frac{\partial \psi^*}{\partial \tau} \frac{dl}{\sqrt{g_{\tau\tau}}} \tag{59}$$

where ψ^* denotes the disturbance (vanishing at infinity) part of the stream function in (50) and the linearizing factor $g_{\tau\tau}$ is given in (2). The contour integration in (59) is performed over the meridional cross-section arc of the torus, namely

$$dl = \frac{a d\beta}{\cosh \tau_0 - \cos \beta} \tag{60}$$

A rather lengthy calculation yields the following expression for the kinetic energy in (59),

$$E = \pi\rho a \sinh \tau_0 \sum_{n=0}^{\infty} \varepsilon_{n0} D_n \{\tfrac{1}{3} P^1_{n-1/2}(\cosh \tau_0[(1 + \cosh^2 \tau_0)\ddot{Q}_{n-1/2}(\cosh \tau_0) - \\ - \cosh \tau_0 \ddot{Q}_{n+1/2}(\cosh \tau_0) - \cosh \tau_0 \ddot{Q}_{n-3/2}(\cosh \tau_0)] - \\ - \sinh^2 \tau_0 \dot{P}^1_{n-1/2}(\cosh \tau_0)\dot{Q}_{n-1/2}(\cosh \tau_0)\} - \\ - \frac{\pi\rho A}{2a} \sum_{n=0}^{\infty} \frac{\varepsilon_{n0} D_n}{(2n-1)} \left[\frac{4 \sinh \tau_0}{(2n+1)} Q^1_{n-1/2}(\cosh \tau_0)\, \dot{P}^1_{n-1/2}(\cosh \tau_0) - \\ - (2n+1)P^1_{n-1/2}(\cosh \tau_0)\, Q_{n-1/2}(\cosh \tau_0)\right] \tag{61}$$

and the added-mass coefficient is then given by

$$\lambda_1 = 2E. \tag{62}$$

It should be noted that (61) and (62) are in fact a closed-form solution for λ_1 which does not require the solution of a tri-diagonal matrix. However, for the numerical computation of λ_1 equation (31) was found to be more efficient and economical than (62) as far as computer time for a prescribed numerical accuracy was concerned.

7. Results and discussion

Equations (31), (41) and (49) were solved numerically for the three coefficients λ_1, λ_2 and λ_6, by employing the Thomas algorithm mentioned before. The numerical results for the three added mass (inertia) coefficients are depicted in Fig. 2 in a dimensionless form where λ_1 and λ_2 are normalized by ρV_T and λ_6 is normalized by ρI_z. Here V_T denotes the volume of the torus and I_z its moment of inertia about the z axis. The various coefficients given in Fig. 2 were calculated for the range $10.1 > d/b > 1.1$. In this range the following fourth-order polynomial approximations are valid:

$$\frac{\lambda_1}{\rho V_T} = 1. + 0.02586527(b/d) + 0.4776554(b/d)^2 - 0.6396613\,(b/d)^3 + 0.181402392(b/d)^4, \tag{63}$$

$$\frac{\lambda_2}{\rho V_T} = 0.5 - 0.0323733\,(b/d) - 0.8639560(b/d)^2 + 1.4155701\,(b/d)^3 - 0.6835380(b/d)^4, \tag{64}$$

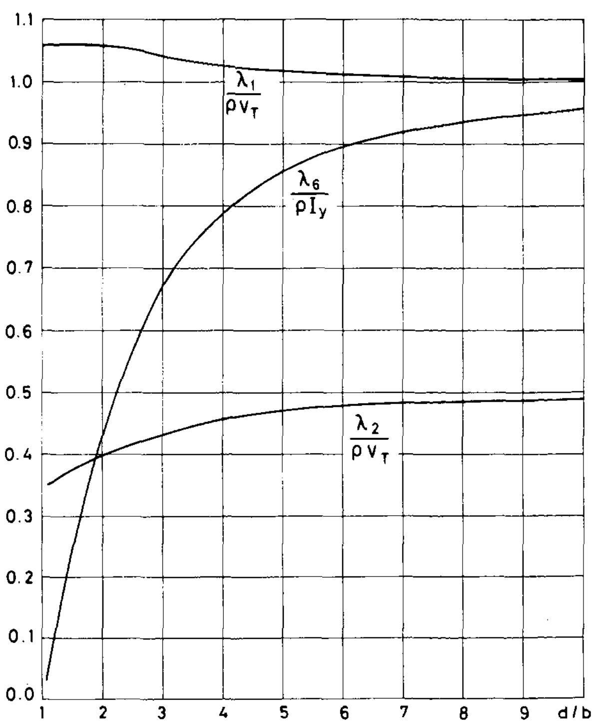

Figure 2. Ratio of generalized added mass to generalized mass of a torus for various pure translatory and rotational motions.

$$\frac{\lambda_6}{\rho I_z} = 1. \; - 0.04220055(b/d) \; - 4.3637653(b/d)^2 + 5.2447998\,(b/d)^3 - 1.8449421(b/d)^4. \qquad (65)$$

Equations (63) and (65) were obtained by fitting a polynomial through the nodal points given by $d/b = 1.1, 2.1, 4.1$ and 7.1. Similarly equation (64) was obtained by using the points $d/b = 1.1$, 3.1, 5.1 and 7.1.

The parameter b/d is always less than 1; the limit $b/d = 0$ corresponds to a slender torus where the core diameter is small compared with the torus diameter. The other extreme value, namely $b/d = 1$ corresponds to the case where the core and the torus radii are equal. Figure 2 shows that the ratio $\lambda_1/\rho V_T$ is approximately unity for all b/d and in fact $1. < \lambda_1/\rho V_T < 1.0625$. In other words, the added mass of a torus, moving normally to the plane of the ring is approximately equal to the mass of the displaced fluid. This is reminiscent of the added-mass coefficient of a two-dimensional cylinder which is exactly unity. This interesting result can be understood by recalling that when $b/d \to 0$ the flow in each cross-section is approximately that around two circles, i.e. parts of two-dimensional cylinders. For a "fuller" torus, i.e. larger value

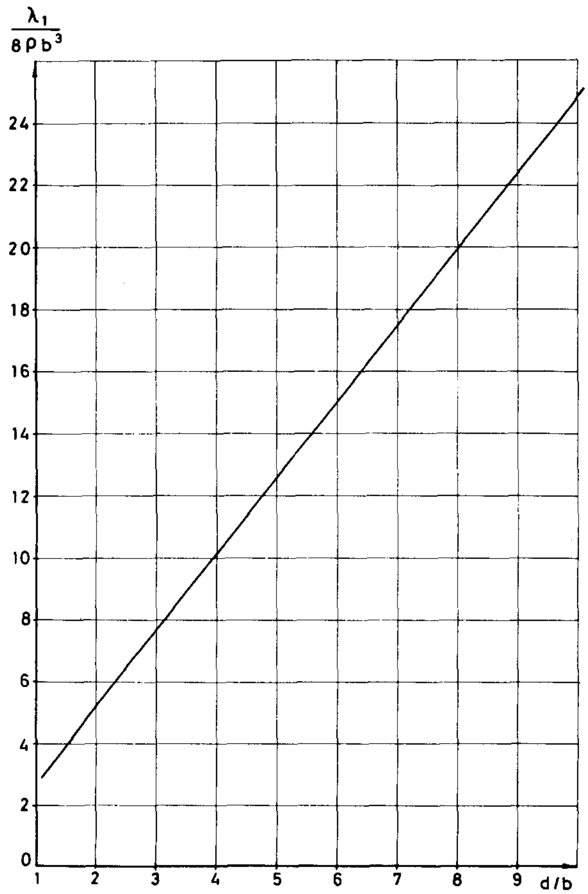

Figure 3. Non-dimensional coefficient of added mass of a torus for pure translation perpendicular to the ring plane, vs. ratio of ring to core diameters.

of b/d, the effect of the finiteness of the core radius is more pronounced and the added-mass coefficient increases. These results are in accordance with the asymptotic analysis of Wu and Yates [12] who show that the added mass of a "very slender" torus is $\lambda_1/\rho V_T = 1$ to first order. Still another useful approximation for non-slender tori is the following linear relation:

$$\frac{\lambda_1}{4\rho d b^2} = 4.96 + 0.264(b/d), \quad 1/10.1 < b/d < 1/1.1 \tag{66}$$

which is also depicted in Figure 3. It should be also mentioned that calculation of λ_1 by means of the stream function method, as given by eq. (62) gave identical results to within accuracy of the numerical calculations. The velocity-potential method (31) was however found to be more advantageous for the computation of λ_1 in terms of computer economy than the stream function method.

The added-mass coefficient for a transverse motion in the plane of the ring, namely $\lambda_2/\rho V_T$, was found to be bounded below by approximately 0.35 for $b/d \to 1$. The upper bound on $\lambda_2/\rho V_T$ was found to be 0.5 reaching this value asymptotically as $b/d \to 0$, as also shown by eq. (64). Again this asymptotic value is in agreement with the approximate results of Wu and Yates

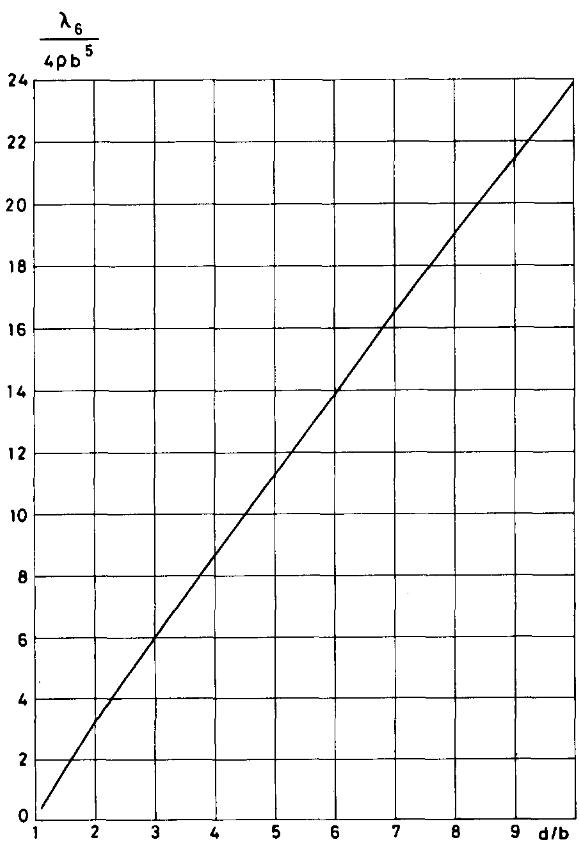

Figure 4. Non-dimensional coefficient of generalized added mass (inertia) of a torus rotating around a ring diameter, vs. ratio of ring to core diameters.

who found that $\lambda_2 = \lambda_1/2$ for very slender tori. The fact that the added mass of a slender torus in the ring plane is approximately one-half can be understood as only about one-half of the sections' velocities are perpendicular to the local section. Again this relation is reminiscent of Munk's relation for the added-mass coefficients of slender bodies of revolution.

The variation of the ratio $\lambda_6/\rho I_z$ versus d/b, as depicted in Fig. 2, shows that for a "full" torus $\lambda_6/\rho I_z$ is very small and that this ratio increases rapidly to the asymptotic value of 1 for slender tori. The "full" torus $b/d \to 1$ resembles a sphere for which $\lambda_6 = 0$, which explains the smallness of the quantity $\lambda_6/\rho I_z$. For a very slender torus the interaction between different cross sections of the torus may be ignored, implying that the added inertia term λ_6 is equal to the moment of inertia ρI_z. Still another useful linear approximation for λ_6 is

$$\frac{\lambda_6}{4\rho b^5} = 2.62\frac{d}{b} - 1.99, \qquad 1.1 \leqq d/b \leqq 10 \tag{67}$$

which is also depicted in Fig. 4.

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